Homework 7 - Sketch of Solutions

#1 For every $f: X \to Y$ there is a commutative diagram

$$\begin{tikzcd} \tilde{H}_0(X) \arrow[r, "\xi"] \arrow[d, "\tilde{f}_*"] & H_0(X) \arrow[d, "f_*"] \\ \tilde{H}_0(Y) \arrow[r, "\xi'"] & H_0(Y) \end{tikzcd}$$

where ξ, ξ' are monos. If $f \simeq g$, $f_* = g_*$. Therefore $\xi' \tilde{f}_* = \xi' \tilde{g}_*$ $\therefore \tilde{f}_* = \tilde{g}_*$

This problem could also be done by using the chain homotopy $\varphi_n : C_n(X) \to C_{n+1}(Y)$.

#3 Show commutativity of the diagram

$$\begin{tikzcd} C_n(X) \arrow[r, "\pi"] \arrow[d, equal] & C_n(X)/C_n(A) \\ C_n(X)/0 \arrow[d, equal] & \\ C_n(X)/C_n(\phi) \arrow[uur, "j_\#"'] & \end{tikzcd}$$

$c \in C_n(X)$, $\quad \pi(c) = c + C_n(A)$

$j_\#(c) = j_\#(c + C_n(\phi)) = c + C_n(A)$.

#4 Let $\Delta : H_1(X,A) \to H_0(A)$ $\quad [z] \in H_1(X,A)$ $\quad z \in C_1(X,A)$

$z = \pi c$, $c \in C_1(X)$ $\quad \partial c = iu$, $\quad [u] = \Delta[z] \in H_0(A) = \frac{C_0(A)}{B_0(A)}$

$\varepsilon_A u = \varepsilon_X iu = \varepsilon_X \partial c = 0$ $\quad \therefore u \in \operatorname{Ker} \varepsilon_A$

Let $\langle u \rangle \in \operatorname{Ker} \varepsilon_A / B_0(A)$ Define $\tilde{\Delta}[z] = \langle u \rangle$ Note

$\xi_A \tilde{\Delta}[z] = \Delta[z]$.

Now define $\tilde{j}_* : \tilde{H}_0(X) \to H_0(X,A)$ by $\tilde{H}_0(X) \xrightarrow{\xi_X} H_0(X) \xrightarrow{j_*} H_0(X,A)$. This defines $\tilde{\Delta}$, $\tilde{j}_*$ and $\tilde{i}_*$ (which was previously defined)

Exactness of at $\tilde{H}_0(A)$

$\tilde{i}_* \tilde{\Delta}[z] = \tilde{i}_* \langle u \rangle = \langle \partial c \rangle = 0$ So $\operatorname{Im} \tilde{\Delta} \subseteq \operatorname{Ker} \tilde{i}_*$

Now suppose $\tilde{i}_*(x) = 0$, $\therefore 0 = \xi_X \tilde{i}_*(x) = i_* \xi_A x$. By exactness (of the unreduced sequence) $\xi_A x = \Delta y = \xi_A \tilde{\Delta} y$ for some y.

$\therefore x = \tilde{\Delta} y$. so $x \in \operatorname{Im} \tilde{\Delta}$.

Exactness at $\tilde{H}_0(X)$

$\tilde{j}_* \tilde{i}_* x = j_* \xi_X \tilde{i}_*(x) = j_* i_* \xi(x) = 0$

$\therefore \operatorname{Im} \tilde{i}_* \subseteq \operatorname{Ker} \tilde{j}_*$.

If $\tilde{j}_* w = 0$, $j_* \xi_X(w) = 0 \quad \therefore \xi_X(w) = i_* v$ some v

Show $v \in \operatorname{Im} \xi_A$ by showing $\varepsilon_{A*} v = 0$. $\therefore v = \xi_A y$

some y. $\xi_X w = i_* v = i_* \xi_A y = \xi_X \tilde{i}_* y$

$\therefore w = \tilde{i}_* y \quad$ so $\quad w \in \operatorname{Im} \tilde{i}_*$.

Exactness at $H_0(X,A)$ i.e. $\tilde{j}_*$ is onto

Let $[x] \in H_0(X,A) \quad x \in C_0(X,A)$, $x = c + C_0(A)$, $c \in C_0(X)$

$\varepsilon_X(c) = n$ for some integer n. Let $a \in C_0(A)$ with $\varepsilon_A a = n$. Then

$x = (c-a) + C_0(A)$, $\varepsilon_X(c-a) = 0$ so $[c-a] \in \tilde{H}_0(X)$

and $\tilde{j}_*[c-a] = [x]$.

#6 $\tilde{H}_i(\mathbb{R}) = 0$ all i. $\therefore$ By the reduced exact sequence of $(\mathbb{R}, \mathbb{Q})$,

$$H_n(\mathbb{R},\mathbb{Q}) \cong \tilde{H}_{n-1}(\mathbb{Q}) \qquad n \geq 1.$$

The path components of $\mathbb{Q}$ are the points of $\mathbb{Q}$. $\therefore \tilde{H}_i(\mathbb{Q}) = 0$

$i > 0$, $\tilde{H}_0(\mathbb{Q})$ is free abelian group on countable set of generators

Also $\quad 0 \to H_1(\mathbb{R},\mathbb{Q}) \to \tilde{H}_0(\mathbb{Q}) \to \underset{\underset{0}{\parallel}}{\tilde{H}_0(\mathbb{R})} \to H_0(\mathbb{R},\mathbb{Q}) \to 0$

$$\therefore H_i(\mathbb{R},\mathbb{Q}) = \begin{cases} 0 & i \neq 1 \\ \text{free abelian on countable set} & i = 1 \end{cases}$$

#7 $Q_n(X) = \bigoplus Q_n(X_\gamma)$

$C_n(X) = \bigoplus C_n(X_\gamma) \qquad A$ is disjoint union of $A \cap X_\gamma$

$C_n(A) = \bigoplus C_n(A \cap X_\gamma)$

$\therefore C_n(X,A) = \bigoplus C_n(X_\gamma, A \cap X_\gamma)$

$\therefore H_n(X,A) = \bigoplus H_n(X_\gamma, A \cap X_\gamma)$.

Consider $H_0(X_\gamma, A \cap X_\gamma)$. If $A \cap X_\gamma = \phi$, $H_0(X_\gamma, A \cap X_\gamma) = \mathbb{Z}$. If ~~$X_\gamma$~~ $A \cap X_\gamma \neq \phi$, use the ~~reduced~~ reduced exact sequence of a pair to conclude $H_0 = 0$.

#9 $f'D + D'g$

#10 $\sum_{i=0}^{n} f^i D g^{n-i}$ chain homotopy ~~btw~~ between f^{n+1} and g^{n+1}

#11 $H_1(E^n, S^{n-1}) \approx \tilde{H}_0(S^{n-1})$ $n \geq 1$

$$H_1(E^n, S^{n-1}) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & n \neq 1 \end{cases}$$

#12 $0 \to Z_n \to C_n \xrightarrow{\partial_n} B_{n-1} \to 0$ is exact and

$B_{n-1} \subseteq C_{n-1}$ is free. ∴ The sequence splits

#13 Show that the chain homotopy $\varphi_n : C_n(X) \to C_{n+1}(Y)$ between $f_\#$ and $g_\#$ induces a chain homotopy between $f_\#, g_\# : C_n(X, A) \to C_n(Y, B)$.

#14 $[z] \in H_{n+1}(X, A)$, $z = \pi(b)$, $\pi : C_{n+1}(X) \to C_{n+1}(X, A)$

$\partial b = i u$, $u \in Z_n(A)$, $\Delta[z] = [u]$

(1) $(f|A)_* \Delta[z] = [(f|A)_\# u]$.

$\Delta' f_* [z] = \Delta' [f_\# z]$ $z = \pi(b)$ ∴ $f_\# z = f_\# \pi(b) =$

$\pi'(f_\# b)$. $\partial' f_\# b = f_\# \partial b = f_\# i_\# u = i'_\# (f|A)_\# u$

∴ (2) $\Delta' f_* [z] = [(f|A)_\# u]$ ∴ (1) = (2).

#15 $f : (X, A) \to (Y, B)$ can be factored as

$$(X, A) \xrightarrow{f'} (B, B) \xrightarrow{j} (Y, B)$$

where f' is f with smaller codomain and j is inclusion.

∴ $f_* = j_* f'_*$. Show (from the axioms) that $H_n(B, B) = 0$, $\forall n$.